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## 順序数の部分空間の有限積の mild normality

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## 概要

The closure of an open set in a topological space is called a regular closed set. A space is called mildly normal (or  $\kappa$ -normal) if every pair of disjoint regular closed sets can be separated by disjoint open sets.

It is known that products of arbitrary many ordinals are mildly normal and products of two subspaces of ordinals are also mildly normal. We characterize the mild normality of products of finitely many subspaces of  $\omega_1$ . Using this characterization, we show that there exist 3 subspaces of  $\omega_1$  whose product is not mildly normal.

A space is said to be *(sub)normal* if every disjoint pair of closed sets is separated by open (resp.  $G_\delta$ -) sets. For a cover  $\mathcal{U} = \langle U_i \mid i \in I \rangle$  of a space, a cover  $\mathcal{V} = \langle V_i \mid i \in I \rangle$  satisfying that  $V_i \subseteq U_i$  for every  $i \in I$  is called a *shrinking* of  $\mathcal{U}$ . A space is said to be *(sub)shrinking* if every open cover has a closed (resp.  $F_\sigma$ -) shrinking. Obviously, every normal (resp. shrinking) space is subnormal (resp. subshrinking). It is easy to see that every (sub)shrinking space is (sub)normal. It is known that every Dowker space is normal, but not subshrinking, so normality does not imply the subshrinking property in general.

Let  $\omega_1$  denote the least uncountable ordinal number.  $C \subseteq \omega_1$  is said to be *cofinal* in  $\omega_1$  if for every  $\alpha < \omega_1$ , there is  $\gamma \in C$  such that  $\alpha \leq \gamma$ .  $C \subseteq \omega_1$  is called a *club* set of  $\omega_1$  if it is closed and cofinal in  $\omega_1$ .  $S \subseteq \omega_1$  is said to be *stationary* in  $\omega_1$  if  $S \cap C \neq \emptyset$  for every club set  $C$  of  $\omega_1$ .

It is known that for  $A, B \subseteq \omega_1$ ,  $A \times B$  is normal iff it is shrinking iff either  $A$  or  $B$  is non-stationary, or  $A \cap B$  is stationary in  $\omega_1$  [8]. Particularly, there are subspaces  $A$  and  $B$  of  $\omega_1$  such that  $A \times B$  is not normal. On the other hand, it is proved in [7] that every subspace of  $\omega_1^2$  is subshrinking, so subnormal. Hence the subshrinking property does not imply normality in general. It was conjectured that every subspace of  $\omega_1^n$  with  $n < \omega$  is subnormal. But we proved in [3] the following theorem.

**THEOREM 1.** (*Hirata, Kemoto [3]*)

- (1)  $\omega_1^3$  has a subspace which is not subnormal.
- (2) Every subspace of  $\{x \in \omega_1^n \mid \forall k_0, k_1 < n (x(k_0) < x(k_1))\}$  with  $n < \omega$  is subshrinking, so subnormal.

The closure of an open set in a topological space is called a *regular closed set*. A space is called *mildly normal* (or  $\kappa$ -normal) if every pair of disjoint regular closed sets can be separated by disjoint open sets. Here we say that a space is *mildly subnormal* if every pair of disjoint regular closed sets can be separated by disjoint  $G_\delta$ -sets. Obviously, every (sub)normal space is mildly (sub)normal. About mild normality, two theorems below were known.

**THEOREM 2.** (*Kalantan, Szeptycki 2002 [6]*)

If  $\alpha_i$  is an ordinal for every  $i \in I$ , then  $\prod_{i \in I} \alpha_i$  is mildly normal.

**THEOREM 3.** (*Kalantan, Kemoto 2003 [5]*)

- (1) For every subspaces  $A$  and  $B$  of ordinals,  $A \times B$  is mildly normal.
- (2)  $(\omega + 1) \times \omega_1$  has a subspace which is not mildly normal.

By the second statement of the theorem above, we can see that the subshrinking property does not always imply mild normality. On the other hand,  $\omega_1 \times (\omega_1 + 1)$  is mildly normal, but not subnormal. Hence mild normality does not imply subnormality in general.

In [5], it is asked whether the product of finitely many subspaces of ordinals are mildly normal. We gave the negative answer for this question. Moreover, we characterized the subshrinking property, subnormality, and mild (sub)normality of products of finitely many subspaces of  $\omega_1$  in terms of stationarity.

**THEOREM 4.** (*Hirata, Kemoto [4], Hirata [2]*)

Let  $\mathcal{A} = \langle A_k \mid k \in N \rangle$  be a finite family of non-empty subspaces of  $\omega_1$  and  $X = \prod_{k \in N} A_k$ . Then the following conditions are equivalent.

- (a)  $X$  is subshrinking.
- (b)  $X$  is subnormal.
- (c)  $X$  is mildly normal.

(d)  $X$  is mildly subnormal.

(e) For every sequence  $\langle k_i \mid i < l \rangle$  of distinct elements of  $N$  with  $2 \leq l \leq |N|$ , if  $A_{k_{i-1}} \cap A_{k_i}$  is stationary in  $\omega_1$  for every  $0 < i < l$ , then  $\bigcap_{i < l} A_{k_i}$  is stationary in  $\omega_1$ .

**COROLLARY 5.** *There are subspaces  $A, B, C$  of  $\omega_1$  such that  $A \times B \times C$  is neither subnormal nor mildly normal.*

*Proof.* Let  $S_0, S_1, S_2$  be disjoint stationary sets of  $\omega_1$ . Put  $A = S_0 \cup S_1$ ,  $B = S_1 \cup S_2$ , and  $C = S_2 \cup S_0$ .  $A \cap B = S_1$  and  $B \cap C = S_2$  are stationary, but  $A \cap B \cap C = \emptyset$ . Hence  $\langle A, B, C \rangle$  does not satisfy the last condition of the theorem.  $\square$

*Proof.* We give here a sketch of a part of the proof of a canonical case of (d)  $\rightarrow$  (e). For the rest part of the proof, see [4] and [2].

Assume that  $\langle A_k \mid k < l \rangle$ ,  $2 \leq l < \omega$ , is a family of subspaces of  $\omega_1$ ,  $A_{k-1} \cap A_k$  is stationary in  $\omega_1$  for every  $0 < k < l$ , and  $\bigcap_{k < l} A_k$  is not stationary in  $\omega_1$ . We will prove that  $X = \prod_{k < l} A_k$  is not mildly subnormal.

Pick a club set  $C$  of  $\omega_1$  disjoint from  $\bigcap_{k < l} A_k$ . Let  $\sigma_0 : l \rightarrow m_0$  and  $\sigma_1 : l \rightarrow m_1$  with  $m_0, m_1 \leq l$  be non-decreasing onto functions such that for each  $0 < k < l$ ,  $\sigma_0(k-1) < \sigma_0(k)$  iff  $k$  is even, and  $\sigma_1(k-1) < \sigma_1(k)$  iff  $k$  is odd. And let  $\tau_0$  and  $\tau_1$  are bijections from  $l$  onto  $l$  such that for each  $i = 0, 1$ ,  $j < m_i$ , and  $k < l$ , if  $\sigma_i^{-1}(\{j\}) = \{k\}$  then  $\tau_i(k) = k$ , and if  $\sigma_i^{-1}(\{j\}) = \{k-1, k\}$  with  $0 < k < l$  then  $\tau_i(k-1) = k$  and  $\tau_i(k) = k-1$ . (The table below expresses values of  $\sigma_i$  and  $\tau_i$  in case  $l = 5$ .)

$k$	$\sigma_0$	$\tau_0$	$\sigma_1$	$\tau_1$
0	0	1	0	0
1	0	0	1	2
2	1	3	1	1
3	1	2	2	4
4	2	4	2	3

For  $k_0, k_1 < l$ , put

$$P(k_0, k_1) = \{x \in X \mid \exists \gamma \in C (x(k_0) \leq \gamma < x(k_1))\},$$

$$E(k_0, k_1) = \{x \in X \mid x(k_0) \leq x(k_1)\}.$$

Then  $P(k_0, k_1)$  is open,  $E(k_0, k_1)$  is closed in  $X$ , and  $P(k_0, k_1) \subseteq E(k_0, k_1)$ . For  $i = 0, 1$ , put

$$P_i = \bigcap \{P(k_0, k_1) \mid k_0, k_1 < l, \tau_i(k_0) < \tau_i(k_1)\},$$

$$E_i = \bigcap \{E(k_0, k_1) \mid k_0, k_1 < l, \tau_i(k_0) < \tau_i(k_1)\}.$$

Then  $P_i$  is open,  $E_i$  is closed in  $X$ , and  $P_i \subseteq E_i$ . Put  $F_i = \text{cl}_X P_i$ . Then  $F_i$  is a regular closed set and  $F_i \subseteq E_i$ . It suffices to show that  $F_0$  and  $F_1$  are disjoint and cannot be separated by disjoint  $G_\delta$ -sets.

Assume that  $0 < k < l$  is odd. Then  $\sigma_0(k-1) = \sigma_0(k)$  and  $\sigma_1(k-1) < \sigma_1(k)$ , so  $\tau_0(k-1) = k > k-1 = \tau_0(k)$  and  $\tau_1(k-1) \leq k-1 < k \leq \tau_1(k)$ . Hence  $E_0 \subseteq E(k, k-1)$  and  $E_1 \subseteq E(k-1, k)$ . In the same way, we have  $E_0 \subseteq E(k-1, k)$  and  $E_1 \subseteq E(k, k-1)$  for every even  $0 < k < l$ . Therefore  $F_0 \cap F_1 \subseteq E_0 \cap E_1 \subseteq \bigcap_{0 < k < l} E(k-1, k) \cap E(k, k-1) = \{\text{const}_l(\alpha) \mid \alpha \in \bigcap_{k < l} A_k\}$  where  $\text{const}_l(\alpha)$  denotes the constant sequence of length  $l$  and of value  $\alpha$ . Let  $\alpha \in \bigcap_{k < l} A_k$  and  $\delta = \sup(C \cap \alpha)$ . (We consider that  $\sup \emptyset = -1$ .) Since  $C$  is closed in  $\omega_1$ ,  $\delta \in C \cup \{-1\}$  holds.  $C$  is disjoint from  $\bigcap_{k < l} A_k$ , so  $\delta < \alpha$  and  $C \cap (\delta, \alpha] = \emptyset$ .  $X \cap (\delta, \alpha]^l$  is a neighborhood of  $\text{const}_l \alpha$  and disjoint from both  $P(1, 0) \supseteq P_0$ . So  $\text{const}_l \alpha \notin F_0$ . Hence  $F_0 \cap F_1 = \emptyset$ .

Let  $G_{i,n}$  be an open set of  $X$  and  $F_i \subseteq G_{i,n}$  for every  $i = 0, 1$  and  $n < \omega$ . We want to see that  $\bigcap_{i=0,1;n<\omega} G_{i,n} \neq \emptyset$ . For  $i = 0, 1$ , let  $\text{pr}_{\sigma_i} : \omega_1^{m_i} \rightarrow \omega_1^l$  denote the mapping such that  $\text{pr}_{\sigma_i}(y) = \langle y(\sigma_i(k)) \mid k < l \rangle$  for every  $y \in \omega_1^{m_i}$ . And let  $Y_i$  be the set of all  $y \in \omega_1^{m_i}$  such that  $\text{pr}_{\sigma_i}(y) \in F_i$  and  $y(j_0) < y(j_1)$  for every  $j_0 < j_1 < m_i$ . It is easy to see that  $Y_i$  is stationary in  $\omega_1^{m_i}$ , that is  $Y_i \cap D^{m_i} \neq \emptyset$  for every club set  $D$  of  $\omega_1$ .

For  $n < \omega$ ,  $G_{i,n}$  is open in  $X$ , so there is a function  $f_{i,n} : Y_i \rightarrow (\omega_1 \cup \{-1\})^{m_i}$  such that  $X_{i,n}(y) \subseteq G_{i,n}$  and  $f_{i,n}(y)(j) < y(j)$  for every  $y \in Y_i$  and  $j < m_i$ , where

$$X_{i,n}(y) = X \cap \Pi_{k < l} (f_{i,n}(y)(\sigma_i(k)), y(\sigma_i(k))).$$

By using Fodor's Pressing Down Lemma generalized by Fleissner, Kemoto, and Terasawa (see [1]), we can pick a stationary tree  $U_{i,n}$  in  $\omega_1^{m_i}$  and a function  $g_{i,n} : \bigcup_{j < m_i} \text{Lv}_j(U_{i,n}) \rightarrow \omega_1 \cup \{-1\}$  such that for every  $u \in \text{Lv}_{m_i}(U_{i,n})$ ,  $u \in Y_i$  and  $f_{i,n}(y) = \langle g_{i,n}(u \upharpoonright j) \mid j < m_i \rangle$  hold. Here

for  $m < \omega$ , we say that  $U$  is a *stationary tree* in  $\omega_1^m$  if there is a family  $\langle \text{Lv}_j(U) \mid j \leq m \rangle$  such that:

- $\text{Lv}_j(U) \subseteq \omega_1^j$  for every  $j \leq m$ ,
- $U = \bigcup_{j \leq m} \text{Lv}_j(U)$ ,
- $u \restriction j' \in \text{Lv}_{j'}(U)$  for every  $j \leq m$ ,  $u \in \text{Lv}_j(U)$ , and  $j' \leq j$ ,
- $\emptyset \in \text{Lv}_0(U)$ ,
- $\text{Move}_U(u) = \langle \alpha < \omega_1 \mid u \restriction \alpha \in U \rangle$  is stationary in  $\omega_1$  for every  $j < m$  and  $u \in \text{Lv}_j(U)$ .

Inductively, we can pick  $x \in \prod_{k < l} A_k$  and  $u_{i,n} \in \text{Lv}_{m_i}(U_{i,n})$  such that  $g_{i,n}(u_{i,n} \restriction j) < x(k) \leq u_{i,n}(j)$  for every  $i = 0, 1$ ,  $j < m_i$ , and  $k < l$  with  $\sigma_i(k) = j$ . For instance, we determine them in case  $n = 5$  in the order  $g_{i,n}(u_{i,n} \restriction \emptyset) = g_{i,n}(\emptyset), x(0), u_{1,n}(0), g_{1,n}(u_{1,n} \restriction 1), x(1), u_{0,n}(0), g_{0,n}(u_{0,n} \restriction 1), x(2), u_{1,n}(1), g_{1,n}(u_{1,n} \restriction 2), x(3), u_{0,n}(1), g_{0,n}(u_{0,n} \restriction 2), x(4), u_{i,n}(2)$ .

$u_{i,n} \in Y_i$ ,  $f_{i,n}(u_{i,n})(\sigma_i(k)) = g_{i,n}(u_{i,n} \restriction j)$ , and  $u_{i,n}(j) = u_{i,n}(\sigma_i(k))$ , so we have  $x \in X_{i,n}(u_{i,n})$  for every  $i = 0, 1$  and  $n < \omega$ . Hence  $x \in \bigcap_{i=0,1;n<\omega} G_{i,n}$ .  $\square$

The problem below is still remained.

**PROBLEM 6.** Let  $\langle A_n \mid n < \omega \rangle$  be a pairwise disjoint family of stationary subspaces of  $\omega_1$ . Is  $\prod_{n<\omega} A_n$  mildly normal?

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